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BERTAND COMPETITION AND THE EXISTENCE OF PURE STRATEGY NASH EQUILIBRIUM IN MARKETS WITH ADVERSE SELECTION

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Bertrand Competition and the Existence of Pure Strategy Nash Equilibrium in Markets with Adverse Selection

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Abstract

I analyse a market with adverse selection in which companies compete à la Bertrand by offering menus of contracts. Contrary to Rothschild and Stiglitz (1976), I allow for any finite number of types and states and more general utility functions. I define the generalised Rothschild-Stiglitz Profile of Actions (RSPA), and I show that, in every possible market, if the RSPA is efficient, it is also a pure strategy Nash equilibrium profile of actions. On the contrary, I show that in markets in which the RSPA is not efficient, preferences admit an expected utility representation with strictly increasing and strictly concave VNM utilities and a weak sorting condition holds, no pure strategy Nash equilibrium exists.

JEL CLASSIFICATION: C62, C72, D86, L13

KEYWORDS: Adverse Selection, Bertrand Competition, Nash Equilibrium

1 MOTIVATION

In their celebrated paper, Rothschild and Stiglitz (1976) (RS) show that an equilibrium may exist or fail to exist in environments with adverse selection. Nonetheless, two significant limitations of their analysis are that, first, no explicit game of competition in the market for contracts is specified, and second, their arguments are, for the most part, diagrammatic and concern a highly stylised insurance market with adverse selection with

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only two possible types and two possible states. Moreover, each potential market entrant is only allowed to offer a single contract. Textbook treatments such as Mas Colell et al. (1995) and Jehle and Reny (2011) examine games in which companies compete à la Bertrand by offering menus of contracts but also focus on the canonical insurance market of RS. The scope of this paper is twofold. First, it extends the results of Mas Colell et al. (1995) and Jehle and Reny (2011) to more general environments with any finite number of types and states. Second, it specifies elements in the space of admissible utility functions that are sufficient for the failure of existence of a pure strategy Nash equilibrium.

The remainder of the paper is organised as follows: In Section 2, I describe the general environment with adverse selection, I analytically characterise competition in menus of contracts, and I state two commonly imposed primitive structural assumptions. I also define a pure strategy Nash equilibrium, efficiency and the famous Rothschild-Stiglitz Profile of Actions (RSPA). In Section 3, I show that when the RSPA is efficient, then a pure strategy Nash equilibrium exists. On the contrary, in environments that satisfy the two stated structural assumptions and when the RSPA is not efficient, a pure strategy Nash equilibrium fails to exist.

2 THE MODEL

■ **The General Environment.** There is a measure one of consumers. Each consumer belongs to one of a finite set of types $\theta = 1, \dots, \Theta$. With some abuse of notation, I denote as Θ the set of types. The share of type θ consumers in the population is λ^θ with $\sum_{\theta=1}^N \lambda^\theta = 1$. There is a finite set of possible (individual) states $\omega = 1, \dots, \Omega$. Uncertainty is purely idiosyncratic, and hence, states occur independently among different consumers. A consumer of type θ has probability f_ω^θ of being in state ω with $\sum_{\omega=1}^\Omega f_\omega^\theta = 1$ for every θ . Let $X \subset \mathbb{R}^\Omega$ denote the set of possible contracts, with representative element x . Assume that X is a compact space. A consumer of type θ has preferences represented by a continuous utility function $U^\theta : X \rightarrow \mathbb{R}$. Let us denote as $x_0 = (0, \dots, 0) \in X$ the status quo contract and as $\underline{U}^\theta$ the status quo utility. The profit of the status quo contract is $\pi^\theta(x_0) = 0$ for every θ . In insurance markets, this corresponds to the consumer remaining uninsured. There exist two symmetric companies in the market $i = 1, 2$. Because I only consider symmetric companies, there is no loss of generality in assuming the existence of only two companies. Adding additional companies increases the notational burden without producing any qualitative differences.¹ If type θ buys contract x from company i , then the latter makes an expected profit $\pi^\theta(x) = -\sum_{\omega=1}^\Omega f_\omega^\theta x_\omega$. This profit does not depend on the identity of the company.

□ **More Special Environments.** In applications, further structure is usually imposed. A common assumption is the following sorting assumption.

ASSUMPTION 1: $\forall x \in X, \pi^1(x) \leq \pi^2(x) \leq \dots \leq \pi^\Theta(x)$, and $\forall x \in X, \theta \in \Theta, \exists x' \in X$ such that $U^\eta(x') > U^\eta(x) \forall \eta \geq \theta$ and $U^\eta(x') < U^\eta(x) \forall \eta < \theta$.

¹Note the direct analogy with traditional Bertrand competition in a market for a homogeneous product, in which the existence of only two firms is sufficient for effective competition.

This assumption imposes considerable structure in the contract space and the space of admissible utility functions. First, the profit of every contract is always increasing in the same order of types. In simple terms, companies prefer higher type consumers to lower type consumers. Second, for every possible contract x and type θ , there exists another contract x' such that every type higher in the rank than θ strictly prefers contract x' over x , and every other type strictly prefers contract x over x' . This assumption is weaker than the usual single crossing condition of Mirrlees (1971) and Spence (1973) because indifference curves may cross more than once but still satisfy Assumption 1.²

The second most frequently imposed assumption states that preferences over contracts for every type admit an expected utility representation with continuous, strictly increasing and strictly concave VNM utility indexes. Formally,

ASSUMPTION 2: $U^\theta(x) = \sum_{\omega=1}^{\Omega} f_{\omega}^{\theta} v_{\omega}(x_{\omega})$, where v_{ω} is continuous, strictly increasing and strictly concave for every $\omega = 1, \dots, \Omega$.

As shown below, this assumption is key to ensuring effective competition between the two companies. Note also that the expected utility index might be state-dependent, unlike most studies in insurance markets in which the utility function is state-independent. Nonetheless, note that within the class of environments with additively separable preferences, those satisfying Assumption 2 are not the most general environments. Indeed, in environments satisfying Assumption 2, consumers differ with respect to their riskiness and other characteristics, such as wealth, but do not differ with respect to their risk-aversion; i.e., the VNM utility index is type independent. When consumers differ with respect to their risk aversion, the use of deterministic contracts is not sufficient to ensure effective competition. This is because type-dependent utility indexes do not permit the use of certainty equivalents, as I essentially do in Theorem 2. In that case, the use of random contracts, i.e., lotteries over the contract space, becomes indispensable. I refrained from using random contracts to avoid overcomplicating the analysis.

□ **Menus, Demands and Profits.** Companies compete à la Bertrand by offering menus of contracts. Each of the two companies selects a menu of contracts. Effectively, the set of possible actions for each company is $2^{|X|}$, where $2^{|X|}$ is the power set of X . Let α_i denote an action for company i and $\alpha = (\alpha_1, \alpha_2)$ a profile of actions.

Based on all contracts that are available in the market, each consumer purchases a unique contract from one of the two companies. Let $x^{\theta}(\alpha)$ denote the contract that maximises the utility of type θ when the action profile is α . Because this might not be unique, assume that this contract is the one that has the maximum profit. Let $d_i^{\theta}(x, \alpha)$ denote the

²Single crossing would further impose that $\forall x \in X, \exists x' \in X$ such that $U^{\eta}(x') > U^{\eta}(x) \forall \eta \leq \theta$ and $U^{\eta}(x') < U^{\eta}(x) \forall \eta > \theta$.

demand for contract x by type θ when the action profile is α . This demand is given by:

$$(2.1) \quad d_i^\theta(x, \alpha) = \begin{cases} 0, & \text{if } x \neq x^\theta(\alpha) \\ \frac{\lambda^\theta}{2}, & \text{if } x = x^\theta(\alpha) \text{ and } x \in \alpha_j \\ \lambda^\theta & \text{if } x = x^\theta(\alpha) \text{ and } x \notin \alpha_j \end{cases}$$

This demand function is analogous to the demand function in a Bertrand market for a homogeneous product. The demand for contract x by type θ is zero when this contract is not the one that maximises the utility of type θ . The demand for contract x is $\frac{\lambda^\theta}{2}$ if this contract maximises the utility of type θ but is also offered by the other company. Finally, the demand for contract x is λ^θ if this contract maximises the utility of type θ but is not offered by the other company. Based on the demand described in (2.1), the expected profit of firm i is written as:

$$\Pi_i(\alpha_i, \alpha_{-i}) = \sum_{\theta \in \Theta} \sum_{x \in \alpha_i} d_i^\theta(x, \alpha) \pi^\theta(x)$$

Companies set their menus independently. In a Nash equilibrium, no firm has a unilateral deviation within the set of feasible actions. A formal definition of Nash equilibrium follows:

DEFINITION 1: A Nash equilibrium consists of a profile of actions $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2)$ such that for every $i = 1, 2$: $\bar{\alpha}_i \in \arg \max_{\alpha_i} \Pi_i(\alpha_i, \bar{\alpha}_{-i})$.

□ **Efficiency.** To characterise the equilibrium set, it is indispensable to define efficiency.

DEFINITION 2: A profile of actions $\alpha = (\alpha_1, \alpha_2)$ is efficient if and only if (i) $\Pi_i(\alpha_i, \alpha_{-i}) \geq 0$ for every i and (ii) there exists no other profile of actions $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2)$ such that $\Pi_i(\tilde{\alpha}_i, \tilde{\alpha}_{-i}) \geq 0$ for every i and $U^\theta(x^\theta(\tilde{\alpha})) \geq U^\theta(x^\theta(\alpha))$ for every θ , with the inequality being strict for at least one θ .

One can easily see that efficiency here is defined with respect to the payoff of consumers. In other words, an efficient profile of actions maximises the utility of all types within the set of profiles that make companies at least break even.

□ **The Rothschild-Stiglitz Profile of Actions.** A profile of actions that plays a special role in markets with adverse selection is what it is usually called the *Rothschild-Stiglitz Profile of Actions* (RSPA).³ A formal definition of a RSPA follows:

DEFINITION 3: A profile of actions $\alpha = (\alpha_1, \alpha_2)$ is an RSPA if and only if: (i) $\pi^\theta(x^\theta(\alpha)) \geq 0$ for every $\theta \in \Theta$ and (ii) there exists no other $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2)$ with $\pi^\theta(x^\theta(\tilde{\alpha})) \geq 0$ for every $\theta \in \Theta$ and $U^\theta(x^\theta(\tilde{\alpha})) \geq U^\theta(x^\theta(\alpha))$, with the inequality being strict for at least one θ .

³In the simple canonical insurance market, this corresponds to the well-known separating pair of contracts.

An RSPA is a profile of actions that maximises the payoff of every type within the set of profiles of actions with the special characteristic that each contract for each of the types makes non-negative profits. By definition, all RSPAs provide the same utility to every type. For convenience, I will assume that there exists a unique RSPA and denote this as α^{RS} .

3 EQUILIBRIA

■ **Existence.** By generalising the intuition of RS, one can prove the following general result:

THEOREM 1: *If the RSPA is efficient, then a Nash equilibrium in pure strategies exists.*

PROOF: Suppose that α^{RS} is efficient. By Definition 2, for every α that dominates α^{RS} , $\Pi_i(\alpha_i, \alpha_{-i}) < 0$ for at least one i . I prove by contradiction that α^{RS} satisfies the definition of equilibrium. Assume not. There exists i and $\tilde{\alpha}_i \neq \alpha_i^{RS}$ such that for some $\Phi \subseteq \Theta$:

- (i) $\max_{x \in \tilde{\alpha}_i} U^\theta(x) > U^\theta(x^\theta(\alpha^{RS})) \forall \theta \in \Phi$
- (ii) $\max_{x \in \tilde{\alpha}_i} U^\theta(x) < U^\theta(x^\theta(\alpha^{RS})) \forall \theta \in \Theta - \Phi$
- (iii) $\Pi_i(\tilde{\alpha}_i, \alpha_{-i}^{RS}) > \Pi_i(\alpha_i^{RS}, \alpha_{-i}^{RS})$

Consider now the symmetric profile of actions $\hat{\alpha} = (\hat{\alpha}, \hat{\alpha})$, where $\hat{\alpha} = ((x^\theta(\alpha^{RS}))_{\theta \in \Phi}, (x^\theta(\tilde{\alpha}))_{\theta \in \Theta - \Phi})$. The profits of the two companies for this profile of actions are equal and given by:

$$\Pi_i(\hat{\alpha}, \hat{\alpha}) = \Pi_{-i}(\hat{\alpha}, \hat{\alpha}) = \frac{1}{2} \left(\sum_{\theta \in \Phi} \lambda^\theta \pi^\theta(x^\theta(\tilde{\alpha}_i, \alpha_{-i}^{RS})) + \sum_{\theta \in \Theta - \Phi} \lambda^\theta \pi^\theta(x^\theta(\alpha^{RS})) \right) > 0$$

which follows from (i)-(iii) and Definition 3. Note, however, that because of (i) and (ii), $\hat{\alpha}$ dominates α^{RS} and has positive profit, contradicting that for every α that dominates α^{RS} , $\Pi_i(\alpha_i, \alpha_{-i}) < 0$ for at least one i . Q.E.D.

Theorem 1 generalises the existence result of RS and Wilson (1977, pp. 186). Note that no assumptions are required to prove this result, and therefore, it is applicable to the general environment.

□ **Non-Existence.** What happens when the RSPA is not efficient? This question is difficult to answer in the general environment. Nonetheless, as I formally prove, Assumptions 1 and 2 are sufficient for the failure of existence of a pure strategy equilibrium, a result that re-enforces the insights of RS.

THEOREM 2: *If Assumptions 1 and 2 are satisfied and the RSPA is efficient, then a Nash equilibrium in pure strategies does not exist.*

PROOF: I prove the result using the following two auxiliary lemmas:

AUXILIARY LEMMA 1: *If Assumption 1 is satisfied and $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2)$ is a pure strategy equilibrium, then $\pi^\theta(x^\theta(\bar{\alpha})) = 0$ for every θ .*

PROOF: I prove the result by contraposition. Suppose that Assumption 1 holds and take $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2)$ such that for some θ , $\pi^\theta(x^\theta(\bar{\alpha})) > 0$. By Assumption 1, there exists a contract x' such that $U^\eta(x') > U^\eta(x^\theta(\bar{\alpha}))$ for every $\eta \geq \theta$ and $U^\eta(x') < U^\eta(x^\theta(\bar{\alpha}))$ for every $\eta \leq \theta$. Consider contract $\tilde{x}$ such that:

$$(3.1) \quad U^\theta(\tilde{x}) = qU^\theta(x^\theta(\bar{\alpha})) + (1 - q)U^\theta(x')$$

$$(3.2) \quad q\pi^\eta(x^\theta(\bar{\alpha})) + (1 - q)\pi^\eta(x') > \frac{1}{2}\pi^\eta(x^\theta(\bar{\alpha}))$$

$$(3.3) \quad 0 \leq q \leq 1$$

Consider now company i and action $\tilde{\alpha}_i$, where $\tilde{\alpha}_i$ includes all of the contracts included in $\bar{\alpha}_i$ plus contract $\tilde{x}$. For any q satisfying (3.1), (3.2) and (3.3), at least type θ buys contract $\tilde{x}$ and the profit of company i from $\tilde{\alpha}_i$ when company $-i$ plays $\bar{\alpha}_{-i}$ is:

$$(3.4) \quad \lambda^\theta \pi^\theta(\tilde{x}) + \sum_{\eta \neq \theta} \sum_{x \in \bar{\alpha}_i} d_i^\eta(x, \alpha) \pi^\eta(x) \leq \Pi_i(\tilde{\alpha}_i, \bar{\alpha}_{-i}) \leq \sum_{\eta=\theta}^\Theta \lambda^\eta \pi^\eta(\tilde{x}) + \sum_{\eta=1}^\Theta \sum_{x \in \bar{\alpha}_i} d_i^\eta(x, \alpha) \pi^\eta(x)$$

where the upper bound of (3.4) is due to Assumption 1 (recall that $\forall x, \pi^1(x) \leq \pi^2(x) \leq \dots \leq \pi^\Theta(x)$). Note, however, that the lower bound of (3.4) is strictly higher than

$$\Pi_i(\bar{\alpha}_i, \bar{\alpha}_{-i}) = \frac{\lambda^\theta}{2} \pi^\theta(\tilde{x}) + \sum_{\eta \neq \theta} \sum_{x \in \bar{\alpha}_i} d_i^\eta(x, \alpha) \pi^\eta(x)$$

, and hence $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2)$ does not satisfy Definition 1. ■

AUXILIARY LEMMA 2: *If Assumption 2 is satisfied and $\bar{\alpha} = (\bar{\alpha}_1, \bar{\alpha}_2)$ is a pure strategy equilibrium, then $\bar{\alpha}$ is efficient.*

PROOF: I prove the result by contraposition. Suppose that $\alpha = (\alpha_1, \alpha_2)$ does not satisfy Definition 2 and suppose that:

$$(3.5) \quad \Pi_i(\alpha_i, \alpha_{-i}) \geq 0 \quad \forall i = 1, 2$$

I will show that α cannot satisfy Definition 1 even if Assumption 2 is satisfied. By Definition 2, there exists $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2)$ such that $\Pi_i(\hat{\alpha}_i, \hat{\alpha}_{-i}) \geq 0$ for every i and $U^\theta(x^\theta(\hat{\alpha})) \geq U^\theta(x^\theta(\alpha))$ for every θ , with the inequality being strict for at least one θ . It is straightforward from (3.5) that there exists j such that $\Pi_j(\alpha_j, \alpha_{-j}) \leq \sum_i \Pi_i(\alpha_i, \alpha_{-i})$. Consider $\tilde{\alpha}_j = (\tilde{x}^\theta)_\theta$, where:

$$(3.6) \quad \tilde{x}_\omega^\theta = qv_\omega(x_\omega^\theta(\alpha)) + (1 - q)v_\omega(x_\omega^\theta(\hat{\alpha})) \text{ for every } \omega$$

Recall that $v_\omega(\cdot)$ is the state utility index and $x_\omega^\theta(\alpha)$ is the transfer that type θ receives (or pays) in state ω when the profile of actions is α . Because $v_\omega(\cdot)$ is strictly concave, $\tilde{x}_\omega^\theta < qx_\omega^\theta(\alpha) + (1-q)x_\omega^\theta(\hat{\alpha})$ by Jensen's inequality, or $-\tilde{x}_\omega^\theta > -(qx_\omega^\theta(\alpha) + (1-q)x_\omega^\theta(\hat{\alpha}))$. Because π^θ is linear, summing up over ω yields: $\pi^\theta(\tilde{x}^\theta) > q\pi^\theta(x^\theta(\alpha)) + (1-q)\pi^\theta(x^\theta(\hat{\alpha}))$. Due to (3.6), $U^\theta(\tilde{x}^\theta) > U^\theta(x^\theta(\alpha))$ for every θ , and therefore:

$$(3.7) \quad \Pi_j(\tilde{\alpha}_j, \alpha_{-j}) = \sum_{\theta} \sum_{x \in \tilde{\alpha}_j} \lambda^\theta \pi^\theta(x)$$

In other words, all consumers in the market are attracted by company j because its menu of contracts is strictly better for all types. Re-write (3.7) as:

$$(3.8) \quad \Pi_j(\tilde{\alpha}_j, \alpha_{-j}) = \sum_{\theta} \lambda^\theta q(\pi^\theta(x^\theta(\alpha)) + (1-q)\pi^\theta(x^\theta(\hat{\alpha})))$$

For any q satisfying

$$\sum_{\theta} \lambda^\theta (q\pi^\theta(x^\theta(\alpha)) + (1-q)\pi^\theta(x^\theta(\hat{\alpha}))) > \sum_{\theta} \sum_{x \in \alpha_j} d_j^\theta(x) \pi^\theta(x)$$

, we have that $\Pi_j(\tilde{\alpha}_j, \alpha_{-j}) > \Pi_j(\alpha_j, \alpha_{-j})$, and hence, $\alpha = (\alpha_1, \alpha_2)$ does not satisfy Definition 1. ■

Now suppose that the α^{RS} is not efficient. Then, for every efficient profile of actions, there exists θ such that $\pi^\theta(x^\theta(\bar{\alpha})) > 0$. Suppose that an equilibrium exists. From Auxiliary Lemma 2, the equilibrium action profile is efficient. This immediately contradicts Auxiliary Lemma 1. Q.E.D.

Why does an equilibrium not exist when Assumptions 1 and 2 are satisfied and the RSPA is not efficient? When the RSPA is not efficient, every efficient profile of actions necessarily involves cross-subsidisation, in the sense that some subset of types subsidises some other subset. As I have argued above, because of Assumption 2, a necessary condition for an equilibrium (if that existed) is for the (equilibrium) profile of actions to be efficient. Hence, if an equilibrium exists, it entails cross-subsidisation. It is then easy to show that, if Assumption 1 is satisfied, there always exists a profitable, cream-skimming deviation by one of the companies. Hence, no profile of actions satisfies the definition of equilibrium.

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