Stochastic modeling of consumer purchase behavior : II Applications

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II. APPLICATIONS

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STOCHASTIC MODELING OF CONSUMER PURCHASE BEHAVIOR:

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This paper tests four alternative composite models of market behavior over a set of consumer panel data for three product categories (margarine, regular coffee and instant coffee). Three of these models are based on various (Condensed) NBD models to describe product purchase distributions. The fourth composite model involves the compound Inverse Gaussian distribution as a purchase timing model. In each case, the beta binomial distribution represents brand choice. The empirical results demonstrate the robustness of the NBD model to departures from its assumptions. The fit provided by the composite model involving the well-known NBD model as a purchase incidence model is best among all alternative models. The gamma distribution seems to give a better description of heterogeneity than the natural conjugate family of distribution for the Inverse Gaussian (IG) distribution. On the other hand, the IG distribution provides an adequate fit to individual interpurchase times. However, the superiority of the fit at the individual level does not offset the lack of adequacy of the model for heterogeneity in purchasing behavior across consumers.

STOCHASTIC MODELING OF CONSUMER PURCHASE BEHAVIOR: II. APPLICATIONS

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Theoretical results on individual and market behavior were derived under various assumptions about product class purchases and brand choice (Bemmaor 1981). Alternative models of product purchase timing were analyzed: the Negative Binomial Distribution, the Condensed-2 Negative Binomial Distribution and the Condensed-3 Negative Binomial Distribution. Another model which was recently proposed as a model of purchase incidence was also studied: the compound Inverse Gaussian distribution (Banerjee and Bhattacharyya 1976). Given its fairly strong empirical support, the heterogeneous zero order model was the only model of brand choice being analyzed. Under the hypothesis of independence between purchase timing and brand choice, aggregate market statistics such as market brand penetration, mean and variance of the number of brand purchases and brand purchase distribution over a fixed time-period were theoretically computed. The purpose of this paper is to test four alternative composite models on a set of SECODIP consumer panel data. The fit of these models is investigated at three distinct levels: 1) modeling of individual consumer interpurchase times, 2) modeling of the variation of individual model parameters over the population, and 3) modeling of market brand choice behavior. Since Herniter's paper (1971), researchers have developed and tested composite purchase timing and brand choice models on consumer panel data. Building upon the work of Chatfield and Goodhardt (1973) who suggested the use of the Condensed-2 NBD as a purchase timing model, Zufryden (1978) and Jeuland, Bass and Wright (1980) added a brand choice model and assessed the fit of the market model to empirical data[1]. Zufryden discussed the use of the linear learning model whereas Jeuland, Bass and Wright applied the Dirichlet model to brand choice. In both cases, they assumed independence between purchase timing and brand choice. In Jeuland, Bass and Wright study, the maximum correlation (in absolute value) between relative frequencies of choice and average purchase rate of the product class was below 0.10. In contrast, Zufryden does not report any empirical result on this hypothesis. This paper follows similar lines as the preceeding papers to the extent that it also combines purchase timing and brand choice models. However, it is relatively more extensive in its scope since several alternative purchase incidence models are tested. Furthermore, it also investigates the fit of the model at the individual level as well as the quality of the predictions of heterogeneity across consumers. Such a basic hypothesis of the composite models as the independence between purchase timing and brand choice has been empirically tested. The first section deals with <u>product</u> purchase models. The second section focuses on the prediction of <u>brand</u> purchase distribution, and <u>brand</u> cumulative penetration over several time periods for the alternative models. The third section is the conclusion.

Data

Consumer panel data for three product categories (margarine, regular coffee and instant coffee) were used as a basis for analysis. These data consist of weekly purchasing records including the date the product was bought, the brand chosen, the package size, the store visited, the price paid and the total quantity bought. Only the households who continuously reported their purchases over a two-year period (1974 and 1975) included in the analysis. These panel data comprise purchases of 1,207 households for margarine, 1,688 households for regular coffee, and 1,257 households for instant coffee. These households bought the product class at least once over the two-year period.

Product Purchase Models

As in any renewal process, two alternative processes might be equivalently investigated, either times between purchases or the number of purchases over a fixed time-interval (Cox and Lewis 1966, p. 78). However, interpurchase times need be independently and identically distributed. Since the variation of interpurchase times might not only be random but also systematic, the assumption of identical distribution does not seem warranted when the quantity bought varies across purchase occasions. The relationship between interpurchase time and quantity bought has been empirically investigated for the three product categories under study.[2]

Test of independence between interpurchase times and quantity bought

We examine the relationship existing between quantity bought and time to next purchase. Consumers are expected to wait longer than normal when large quantities are bought. The testing procedure is now detailed. For each product category, a cutoff quantity Q_0 has been selected. This quantity approximately corresponds to the median quantity purchased on a single purchase occasion for a relatively large number of households. An average interpurchase time corresponding to quantities less than Q_0 has been computed and compared to the average interpurchase time for quantities greater than or equal to Q_0 for each household under investigation. Two testing procedures have been used: a) parametric test of equality of means across households, and b) nonparametric test (sign test) of equality of means across households. By using the first test, we attempt to assess the magnitude of the differences between means whereas the second type of testing procedure provides information on the consistency of the relationship. The results of these tests are shown in Table 1. Interpurchase times strongly depend on quantity bought for all three product categories. Note that the relationship between quantity bought and interpurchase time seems less strong for margarine than for regular coffee and instant coffee. Margarine has a shorter shelf life than coffee; therefore less stocking is done by consumers. In order to circumvent the dependence relationship between quantity bought and interpurchase times, interpurchase times have been rescaled into usage times per unit quantity. As shown in Figure 1.A, this rescaling consists of dividing interpurchase times by quantity previously bought. The purchase pattern depicted in Figure 1.A is transformed into that shown in Figure 1.B. The transformed stochastic process is now assumed to be a renewal process.

Over a fixed time-period, the random variable then becomes the number of unit quantities bought.

Analysis of individual product purchases

Probably, the most flexible model for interpurchase time has been first suggested by Banerjee and Bhattacharyya (1976): it is the two parameter Inverse Gaussian (IG) distribution. Its advantage over the Erlang model is that it may be compounded with a bivariate natural conjugate family of distributions (Ehrenberg 1959, Chatfield and Goodhardt 1973, Bemmaor 1981). The Erlang model only allows for variation in the scale parameter over the population, the shape parameter being constrained to equality across consumers. Bemmaor (1981) showed that all four models belong to the same family of probability distributions: the Generalized IG distribution. The probability density function of the two-parameter IG distribution is given by

$$f(t|\psi,\phi) = (\phi/2\Pi)^{1/2}\ t^{-3/2}\ \exp\{-\phi t(\psi-1/t)^2/2\},$$

$$t>0,\ \psi>0,\ \text{and}\ \phi>0.$$

The mean and variance of the IG(ψ,ϕ) distribution are respectively given by ψ^{-1} and $\psi^{-3}\ \phi^{-1}$. The parameter ψ can be interpreted as the average number of occurrences per unit time. Since the interpurchase times have been rescaled for all three product categories, one occurrence corresponds to one purchase of unit quantity. The location of the mode is $t_0= 1/\psi\{(1+9/4\phi^2\psi^2)^{1/2} - 3/2\phi\psi\}$. Letting $v_0=\phi\psi$, the mode becomes

$t_0=1/\psi\{(1+9/4v_0^2)^{1/2}-3/2\ v_0\}$. The distribution is unimodal and its shape depends on the value of ϕ only (Johnson and Kotz 1970a, p. 137-153). As shown by Tweedie (1957), the maximum likelihood estimators (MLE) of ψ and ϕ are given by

$$\hat{\psi} = 1/\bar{t} \quad \text{and} \quad \hat{\phi} = 1/(\bar{t}\ \bar{t}_r - 1)$$

where $\bar{t}$ and $\bar{t}_r$ denote the mean usage time per unit quantity and the mean of their reciprocals respectively. The MLE of the mean ψ^{-1} is $\bar{t}$. As a conclusion on the fit of the model requires a fairly large number of observations, we have studied the purchasing behavior of the households who purchased each product class more than eighty times over 1974 and 1975. As an example of the quality of the fit to regular coffee purchases, actual and theoretical interpurchase time frequencies for four households are illustrated in Figure 2. The chi-square goodness-of-fit measures are statistically significant in each case. But the model seems to capture the complexity of individual purchasing behavior with a fair degree of approximation, given the small number of parameters. An alternative model of interpurchase times is the Erlang-n distribution with scale parameter λ. Its probability density function is given by

$$f(t|n,\lambda) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t}, \quad t>0,\ n \text{ positive integer and } \lambda>0.$$

The mean and variance are n/λ and n/λ^2 respectively. Since the shape parameter n is constrained for compounding purposes, its fit was expected to be of lower quality than that the IG distribution, the estimator of the ψ parameter of the IG distribution being analogous to that of the parameter λ/n of the Erlang-n distribution. For n known, the MLE of λ/n is $1/\bar{t}$ as that of ψ (Johnson and Kotz 1970b, p. 190). Therefore the MLE of the mean n/λ is also $\bar{t}$.

In order to investigate the magnitude of the order n, the distribution of the coefficients of variation (CR=standard deviation/mean) of interpurchase time distributions was analyzed. For the Erlang-n distribution, the CR is $1/\sqrt{n}$. Table 2 shows the relative frequencies of the CR's over the population for each product class. Note that for margarine and instant coffee, the median is 0.71 ($n \approx 2$) which is consistent with the results of Chatfield and Goodhardt (1973, Table 3) on washing-up liquids, razor blades, dentifrice and toilet soap [3]. On the other hand, for regular coffee, the median is 0.58 ($n \approx 3$) which corroborates the results found by Cynthia Fraser of Columbia University in her doctoral thesis. Regular coffee is more regularly bought than the two other product categories studied. Hence the distribution of the number of purchases over a fixed time-period seems more consistent with a Condensed-3 Poisson than with a Condensed-2 Poisson.

Modeling heterogeneity across consumers

When interpurchase times are distributed IG(ψ,ϕ), Banerjee and Bhattacharyya (1976) assume that ψ and ϕ follow a bivariate distribution function whose probability density function is

$$p(\psi,\phi|\alpha,\beta,\gamma)=(\beta/\alpha)^{1/2}(\gamma\alpha/2)^{\gamma/2}[H_{\nu}(\xi)B(\nu/2,1/2)$$

$$\Gamma(\gamma/2)]^{-1}\ \exp\{-\gamma\alpha/2[1+\beta/\alpha(\psi-1/\beta)^{2}]\phi\}\phi^{(\gamma/2)-1}$$

where $\alpha>0$, $\beta>0$, $\gamma>1$, $\nu=\gamma-1$ and $\xi=(\alpha\beta/\nu)^{-1/2}$. $H_{\nu}(\cdot)$ is the cumulative distribution function of the Student's t-distribution with ν degrees of freedom, and $B(\cdot,\cdot)$ is the beta function. The marginal probability density function of ψ is given by

$$p(\psi|\alpha,\beta,\gamma)=(\beta/\alpha)^{1/2}\ [H_{\nu}(\xi)B(\nu/2,1/2)]^{-1}\ [1+\beta/\alpha(\psi-1/\beta)^{2}]^{-(\nu+1)/2},\ \psi>0. \quad (1)$$

This is the left truncated Student's t-distribution with location parameter β^{-1}, scale parameter $\alpha/(\nu\beta)$, and degrees of freedom ν. The mode of this distribution is at $\psi_{0}=1/\beta$. The marginal probability density function of ϕ is

$$p(\phi|\alpha,\beta,\gamma)=(\gamma\alpha/2)^{\nu/2}\ [H_{\nu}(\xi)\ \Gamma(\nu/2)]^{-1}\ g(z)\ \exp(-\gamma\alpha\phi/2)\phi^{(\nu/2)-1},\ \phi>0 \quad (2)$$

where $g(\cdot)$ is the cumulative standard normal distribution and $z=(\gamma\phi/\beta)^{1/2}$. This is a modified gamma distribution with parameter $\gamma\alpha/2$, $\nu/2$ and γ/β. Alternatively when interpurchase times are assumed to be Erlang-n with scale parameter λ, the mean number of purchases λ/n is distributed gamma (r,σ) over the population. It follows that λ is distributed gamma with parameter $(r,n\sigma)$.

Empirical tests

Once the maximum likelihood estimates of ψ for the IG distribution (or equivalently of λ/n for the Erlang-n distribution) have been computed, the theoretical models for heterogeneity have been parameterized by maximum likelihood through the use of a numerical search procedure based on the generalized reduced gradient method (Gabriele and Ragsdell 1976). Parameters have been estimated over the two-year period for all three product categories. The empirical and fitted left truncated Student's-t distributions of $\hat{\psi}$ are shown in Figure 3. Also, gamma distributions have been fitted to these empirical distributions. The comparison of the magnitude of the χ^2 goodness-of-fit measures indicates that, for each product category, the gamma distribution provides a better fit than the Student's-t distribution. Note, however, that the p-levels (which measure the probability that the χ^2 statistics take on a larger value than that observed, the model under study being true) are still very small for the gamma distribution (less than 10^{-2} in each case). Similarly, the fit of the modified gamma distribution to the relative frequencies of the estimated ϕ values is poor as shown in Figure 4. Overall the empirical results show that the IG dominates the Erlang-n distribution at the individual level, but this advantage seems lost when modeling heterogeneity across consumers. The third step consists of analyzing the fit of the alternative models to aggregate purchasing behavior.

Modeling aggregate product purchase behavior

The compound (Condensed) Poisson distributions have been analytically derived by Ehrenberg (1959), Chatfield and Goodhardt (1973), and Bemmaor (1981). These distributions give the theoretical relative frequencies of the number of product purchases over the population. For the compound IG distribution, Banerjee and Bhattacharyya (1976) suggest the use of the following formula

$$P(k|T,\alpha,\beta,\gamma) = F_{W_k}(T) - F_{W_{k+1}}(T), \qquad k=0,1,2,\ldots \tag{3}$$

where k is the number of product purchases over a time-period of length T in weeks (T=4,8,...,104), W_k is the waiting time to the kth purchase, and $F_{W_k}(\cdot)$ is the cumulative distribution function of W_k over the population. This cumulative distribution function whose probability density function is given in Banerjee and Bhattacharyya (1976, Appendix) has been computed through the use of numerical integration over a rectangular region (NI 1976). The fit of the four distributions over two time-periods is illustrated in Table 3. Note that all four models <u>under</u>estimate the proportion of non-buyers of the product class. The discrepancy is all the larger as the NBD is more condensed. When testing the NBD model, Schmittlein and Morrison (undated) found a similar result on several product classes (peanut butter, garbage bags, food bags and lawn bags). The "shelving" effect already evidenced in Ehrenberg (1959) and Jeuland, Bass and Whright (1980, Table IV) is apparent. A systematic discrepancy between actual and theoretical proportion of buyers occurs for purchasing frequencies of one unit quantity per week (k=4 for one month and k=8 for two months). The models consistently underestimate the proportion of regular buyers. The χ^2 goodness-of-fit measures of all four models are compared in Table 4. In each case, the NBD provides a better fit than the alternative models. The quality of the fit decreases with an increase in the condensing. The compound IG model provides the worst fit among all four alternative models.

Overall these results show that the NBD model, despite its fairly unrealistic assumption of exponential interpurchase times, provides the best fit among all four alternative models. Because of its relatively poor description of heterogeneity across the population, the compound IG distribution does not accurately summarize the product purchasing process.

Brand purchase models

Test of independence between purchase timing and brand choice

The theoretical results on brand purchase timing derived in Bemmaor (1981) are based on the hypothesis of independence between brand choice probabilities and average product class purchasing frequency. This hypothesis was tested on a total of twenty-one brands belonging to the three product classes under study. The simple pairwise correlations between relative frequencies of choice and quantity bought are shown in Table 5. Seven coefficients out of twenty-one are significant at the .05 level. Additional tests involving the use of parametric as well as non parametric procedures were also carried out. They also showed that the assumption of independence seems fairly well grounded for most brands. This result is consistent with those found by Jeuland, Bass and Wright (1980) on catsup and cooking oil. Shoemaker et al. (1977) analysis of instant coffee, regular coffee and paper towels leads to the conclusion that, for thirteen out of fifteen brands, "the independence assumption is a good first approximation".

Estimation of the brand choice model

Assume that the vector of brand choice probabilities over (N+1) brands $(\theta_1,\theta_2,\ldots,\theta_N)$ follows a Dirichlet distribution with parameter vector $(a_1,a_2,\ldots,a_{N+1})$, $a_i>0$ for all i and $\sum_{i=1}^{N+1} a_i=c$. Let the first and second sample moments about the origin for the choice probabilities be

$$m_{i1} = 1/s \sum_{\ell=1}^{s} \hat{\theta}_{i\ell}$$

and

$$m_{i2} = 1/s \sum_{\ell=1}^{s} \hat{\theta}^2_{i\ell}, \qquad i=1,2,\ldots,(N+1).$$

s being the sample size and $\hat{\theta}_{i\ell}$ being the MLE of the probability of choosing Brand i for consumer ℓ. Following Fielitz and Myers (1975), the method of moment parameter estimators are

$$\hat{a}_i = (m_{11}-m_{12})m_{i1}/(m_{12}-m_{11}{}^2) \qquad i=1,\ldots,N$$

and

$$\hat{a}_{N+1} = (m_{11}-m_{12}) \; (1 - \sum_{i=1}^{N} m_{i1})/(m_{12}-m_{11}{}^2).$$

These estimators are consistent but not generally efficient (Rao 1973, p. 351) Contrary to alternative estimators such as maximum likelihood or minimum chi-square estimators, these method of moment estimators have a closed-form expression and, consequently, are simple to compute. The estimates have been computed over the first seven-month period (1 month = four weeks) for the regular coffee brands, and over the entire two-year period for the margarine brands and instant coffee brands.

Fit to the empirical brand purchase distribution

Once the parameters of the Dirichlet model have been estimated, we might compute the brand purchase distribution by unconditioning the brand choice model $P(x|k)$, x being the number of brand purchases, over k product purchases

$$P(x) = \sum_{k=0}^{\infty} P(x|k) \; P(k|r,\sigma) \qquad x=0,1,2,\ldots \qquad (4)$$

In our case, $P(x|k)$ is the beta binomial distribution and $P(k|r,\sigma)$ is one of the four product purchase models[4]. Bemmaor (1981) showed that equation (3) takes a closed-form expression when $P(k|r,\sigma)$ is $NBD(r,\sigma)$. Closed-forms expressions have also been derived for $x=0$ when $P(k|r,\sigma)$ is the Condensed-2 NBD or the Condensed-3 NBD. The fit of the brand purchase distribution (4) has been assessed for three major brands belonging to distinct product categories over two time-periods. In order to study the robustness of the models to departure from their assumptions, we have selected the margarine brand and the regular coffee brand for which purchase timing and brand choice were <u>most</u> correlated. As shown in Table 5, both of these brands tend to be bought by light buyers. On the other hand, the correlation was lowest for the instant coffee brand. Since the NBD model provides the best fit to the product purchase data, and the compound IG model is the worst model, only these two models have been fitted to the brand purchase distribution. For each composite model, the brand choice model is the beta binomial model. As shown in Table 6, the results demonstrate the superiority of the NBD-based model over the compound IG-based model. Despite the rejection of the hypotheses of independence between purchase timing and brand choice for two brands out of three, the NBD-based model provides reasonable predictions. Both models underestimate the proportion of non-buyers (except in one case). Also, they underestimate the proportion of heavy buyers. This lack of spread of the theoretical distribution is evidenced in Table 7 where actual and theoretical coefficient of variation (standard deviation/mean) are compared across the three models involving the (Condensed) NBD as a purchase timing model. Note that the discrepancy between actual and theoretical coefficients of variation increases as the NBD becomes more condensed.

Fit to the brand cumulative penetration

As a further testing of the alternative models, empirical and theoretical cumulative penetrations were compared over several time-periods for two composite models: the NBD-based model and the compound IG-based model. The reason for selecting both of these models is that the former provides the best fit to the empirical brand purchase distribution whereas the latter seems worst. We would expect that the other model predictions would lie in between those two models. Theoretical penetration was computed through the use of numerical integration routine for the NBD-based models (IMSL 1979). For the compound IG-based model, $P(r|k,\sigma)$ was replaced by equation (3) in (4). Given the relatively short period of analysis (two years) by comparison with the average interpurchase time for each product category (more than two weeks), equation (4) converges rather rapidly. Over the two-year period, the compound IG-based model seems to fit the empirical data slightly better than the other model (for two brands out of three). However, the slight improvement does not seem to warrant the inclusion of an additional parameter (three parameters of the compound IG distribution versus two parameters for the NBD distribution).

Conclusion

In this paper, four alternative composite models of market behavior were tested on consumer panel data for three product categories: margarine, regular coffee and instant coffee. Three of these models involve the (Condensed) NBD model as a purchase incidence model whereas the fourth one assumes a compound IG distribution. All of the models integrate the beta binomial model as a model of brand choice.

The empirical results demonstrate the superiority of the model based on the well-known NBD distribution over the three alternative models. These findings corroborate the results of the study by Chatfield and Goodhardt (1973), Schmittlein and Morrison (undated), and Bemmaor and Morrison (1981). They point out the robustness of the NBD model to departures from its basic assumptions, whether it be the Poisson assumption, the gamma mixture or the independence between family size and purchase timing. This characterization of the NBD model extends to the hypothesis of independence between purchase timing and brand choice. These results would deserve being tested over new data sets. For the compound IG model, the adequacy of the fit to individual interpurchase times does not seem to offset the lack of fit to individual parameter distributions over the population. This study opens new avenues for research. It would be useful to integrate decision variables into a stochastic market model framework. The advantage of an NBD-based model by comparison with alternative models is that its statistical properties are simple. In particular, the distribution of the number of product purchases over a fixed time-period being Poisson for an individual consumer, the distribution of brand purchases is also Poisson under the hypotheses of constant probabilities of choice and independence between consecutive choice occasions. Poisson regression models including marketing mix variables might then be developed and tested. Such a research line needs further investigation.

FOOTNOTES

[1] Chatfield and Goodhardt (1973) only refer to the Condensed NBD. We call it here Condensed-2 NBD to distinguish it from other Condensed NBD distributions such as the Condensed-3 NBD that will be tested in the following section.

[2] The hypothesis of independence between brand bought and time-to-next purchase was also empirically investigated. We might expect that for a product category such as instant coffee, consumers take less time to use a given quantity of certain brands because of their lower content of cafeine. A few brands are made of a mixture of chicory and cafeine and others are 100% chicory. Parametric as well as nonparametric tests were used to test differences between mean interpurchase times for various brands across the sample. The results which are not reported here because of space limitations showed that for regular coffee and margarine, the hypothesis of independence was consistent with the evidence for a total of thirty-four pairs of brands studied. For instant coffee, there appeared to be a systematic difference in usage time between two brands only out of a total of twenty seven pairs of brands. As expected, one of these two brands is a mixture coffee/ chicory and the other one is a normal instant coffee brand. Consumers tend to use larger quantities of the mixture per cup so as to enhance the coffee flavor.

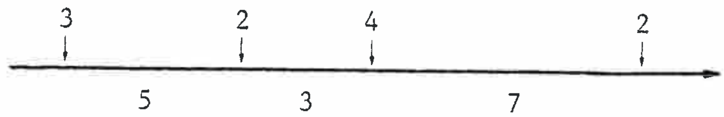
[3] Chatfield and Goodhardt (1973) also report empirical evidence of n close to 2 for times-to-next-purchase of a detergent <u>brand</u>. This is a somewhat weaker support of an Erlang-2 model than the other product categories: there is not a one-to-one correspondence between <u>product</u> purchase and times between <u>brand</u> purchases. In particular, when times between product purchases are Erlang-2, times between <u>brand</u> purchases are <u>not</u> Erlang-2.

[4] When the purchase timing model is the compound IG distribution, we replace $P(k|r,\sigma)$ in (4) by $P(k|\alpha,\beta,\gamma)$.

A. <u>Original pattern</u>

Quantity bought[a]
(in units)

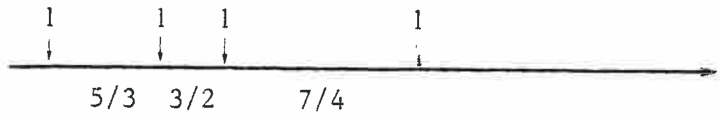
Interpurchase time
(in weeks)



B. <u>Transformed data</u>

Quantity bought
(in units)

Interpurchase time
(in weeks)



[a] *One unit equals 250 grams for margarine and regular coffee, and 50 grams for instant coffee.*

<u>Figure 1</u> Transformation of Interpurchase Times for Individual Consumers

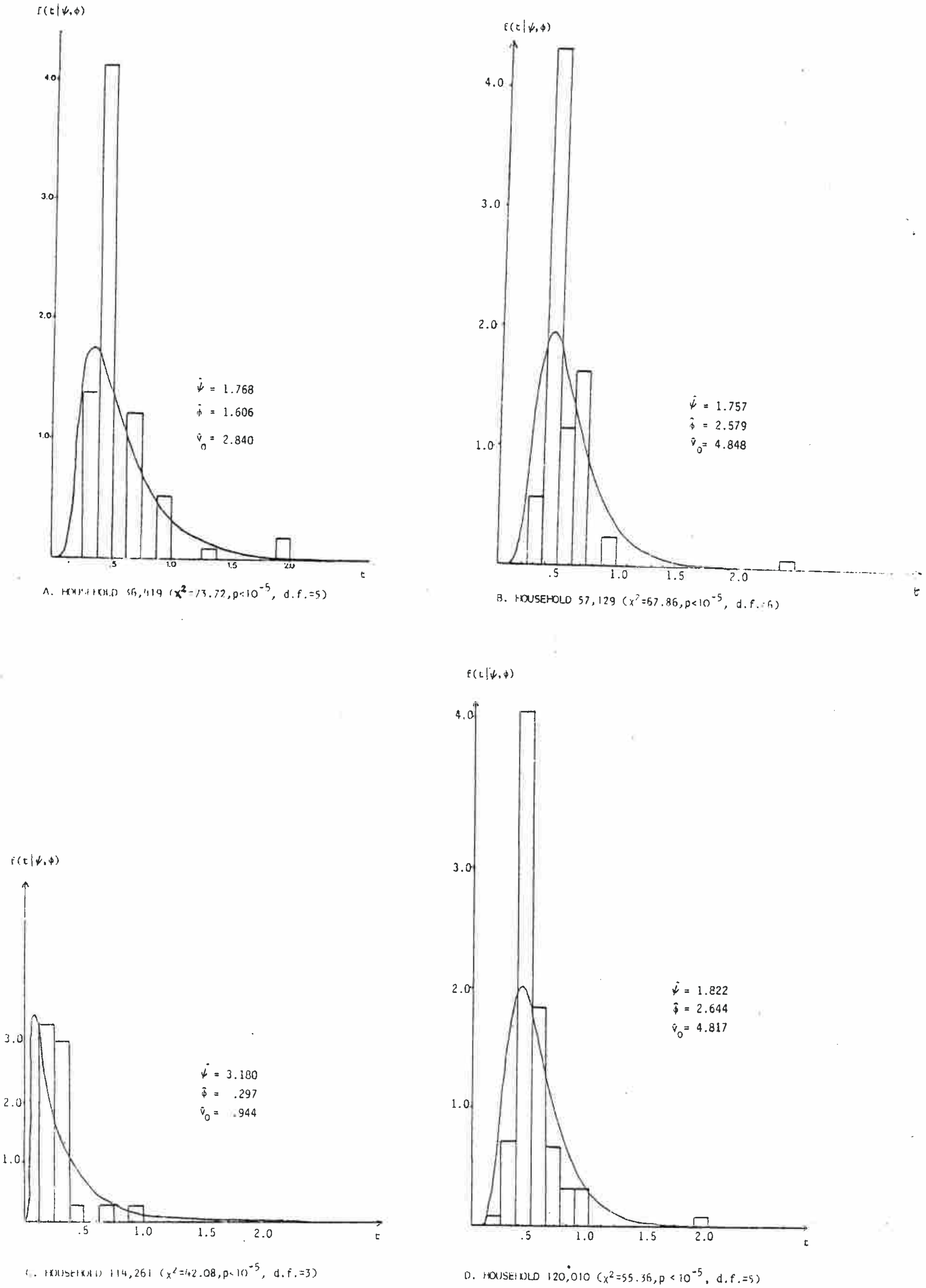
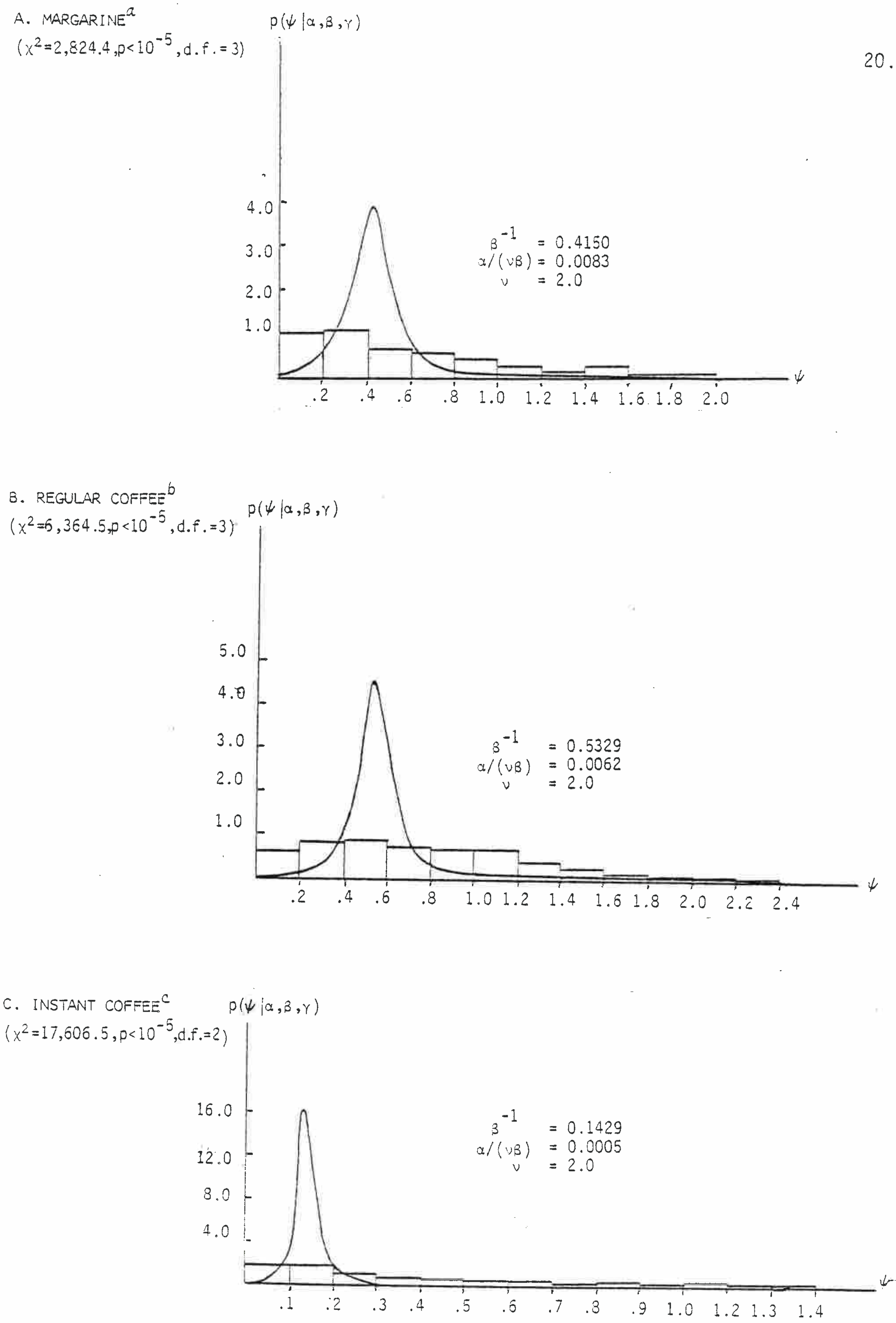


Figure 2 Empirical and Fitted Distributions of Usage Times for Regular Coffee[a]

[a]*One unit equals 250 grams.*

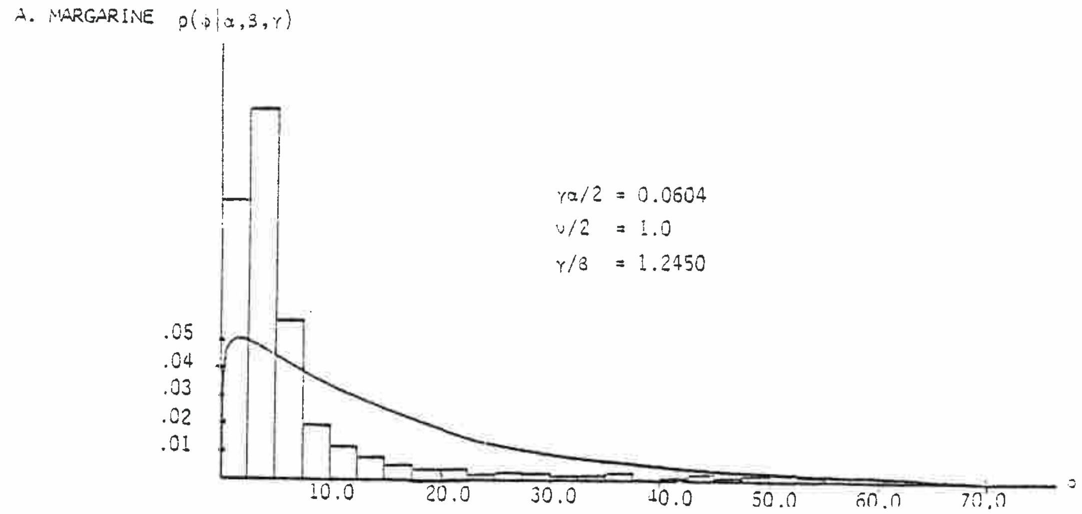


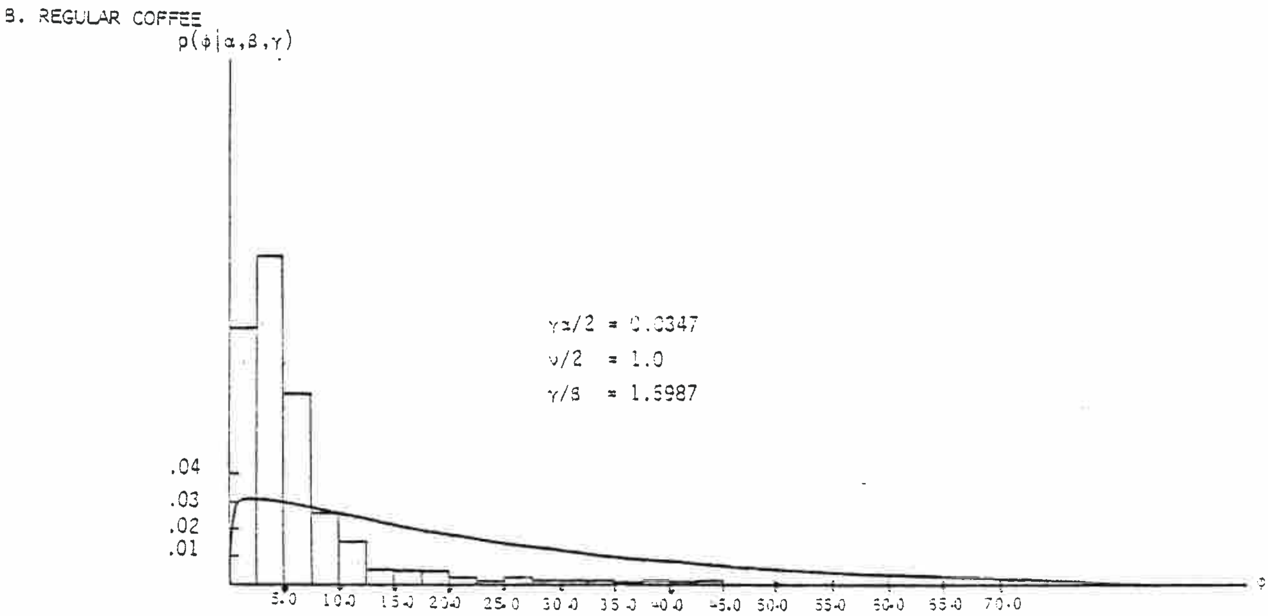
[a] The χ^2 goodness-of-fit measure of the gamma (1.297,.515) distribution is 12.1, $p<.00738$, d.f.=3

[b] The χ^2 goodness-of-fit measure of the gamma (1.781,.412) distribution is 32.3, $p<10^{-5}$, d.f.=3

[c] The χ^2 goodness-of-fit measure of the gamma (1.049,.495) distribution is 132.0, $p<10^{-5}$, d.f.=2.

Figure 3 Empirical and Fitted Left Truncated Student's-t Distributions of ψ





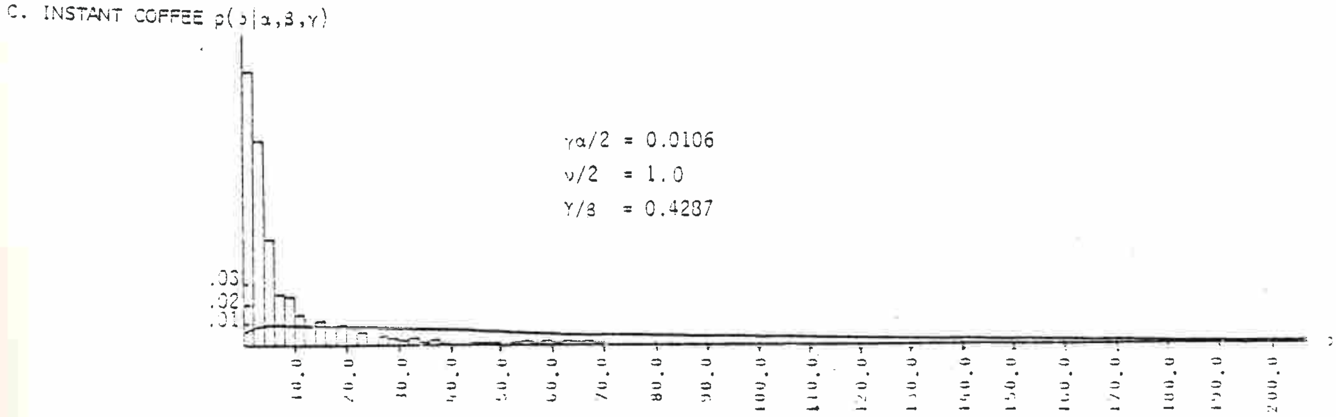


Figure 4 Empirical and Fitted Modified Gamma Distributions of ϕ

Table 1

Test of Independence between Interpurchase Times and Quantity Bought

	Margarine		Regular coffee		Instant coffee	
	Q[a]<500g	Q≥500g	Q<500g	Q≥500g	Q<200g	Q≥200g
Average quantities bought (in grams)	258.025	578.676	267.235	665.541	66.842	239.378
Average inter-purchase times (in weeks)	4.340	4.878	2.961	4.108	7.193	9.316
Standard deviation	5.743	5.467	3.193	4.901	8.506	9.175
z-test	2.272[b]		9.748[c]		4.428[c]	
Sign test	9.140[c]		18.715[c]		8.640[c]	
Base	681		1,343		495	

[a] *Quantity bought*

[b] $p<.012$

[c] $p<5\times10^{-6}$

Table 2

Empirical Distributions of the Coefficients of Variation of Interpurchase Time Distributions[a]

Range	Relative frequencies (in %)		
	Margarine	Regular coffee	Instant coffee
Less than .33	4.2	4.8	7.4
.33 - .35	.9	2.1	2.2
.35 - .38	1.3	1.7	2.0
.38 - .41	1.9	2.5	1.6
.41 - .45	2.6	3.9	2.9
.45 - .50	6.1	6.8	5.2
.50 - .58	11.0	12.9	11.1
.58 - .71	20.6	21.6	17.0
.71 -1.00	30.3	29.4	32.2
More than 1.00	21.2	14.3	18.4
		Median	
	.71	.58	.71

[a] *The coefficient of variation was estimated for the households who made at least three purchases over the two-year period and whose usage rates (quantity bought/interpurchase time) varied over the two-year period: 1,071 households for margarine, 1,267 households for regular coffee and 951 for instant coffee.*

Table 3

Empirical and Fitted Distributions of the Quantity of Use for Each Product Category Over Two Time-Periods[a]

Margarine

k^b	1					2				
0	.379	(.234)[c]	(.183)[d]	(.165)[e]	(.137)[f]	.277	(.120)[c]	(.088)[d]	(.078)[e]	(.044)[f]
1	.110	(.204)	(.230)	(.248)	(.493)	.071	(.126)	(.131)	(.140)	(.077)
2	.169	(.158)	(.178)	(.186)	(.270)	.120	(.116)	(.124)	(.127)	(.225)
3	.051	(.117)	(.129)	(.131)	(.060)	.044	(.103)	(.110)	(.113)	(.337)
4	.104	(.085)	(.090)	(.090)	(.019)	.097	(.089)	(.094)	(.098)	(.180)
5	.020	(.061)	(.062)	(.061)	(.011)	.036	(.076)	(.080)	(.081)	(.068)
6	.056	(.043)	(.042)	(.041)	(.002)	.077	(.064)	(.067)	(.069)	(.029)
7	.009	(.030)	(.028)	(.027)	(.002)	.072	(.054)	(.056)	(.056)	(.015)
8	.047	(.021)	(.020)	(.018)	(.001)	.050	(.045)	(.046)	(.046)	(.008)
9	.000	(.015)	(.012)	(.012)	(.001)	.015	(.037)	(.038)	(.039)	(.005)
10+	.055	(.032)	(.026)	(.021)	(.004)	.191	(.170)	(.166)	(.153)	(.012)
Median	2	(2)	(2)	(2)	(1)	3	(4)	(4)	(4)	(3)

Regular Coffee

k^b	1					2				
0	.218	(.177)[g]	(.126)[h]	(.107)[i]	(.048)[j]	.133	(.075)[g]	(.048)[h]	(.039)[i]	(.018)[j]
1	.118	(.196)	(.212)	(.218)	(.320)	.061	(.102)	(.099)	(.096)	(.023)
2	.139	(.169)	(.191)	(.201)	(.508)	.074	(.109)	(.113)	(.115)	(.073)
3	.103	(.133)	(.148)	(.156)	(.089)	.065	(.105)	(.113)	(.115)	(.248)
4	.168	(.099)	(.106)	(.108)	(.020)	.096	(.096)	(.104)	(.108)	(.393)
5	.066	(.071)	(.074)	(.076)	(.009)	.059	(.086)	(.092)	(.093)	(.153)
6	.055	(.050)	(.050)	(.049)	(.0[illegible]2)	.078	(.074)	(.078)	(.081)	(.049)
7	.026	(.035)	(.032)	(.032)	(.002)	.052	(.063)	(.066)	(.069)	(.018)
8	.052	(.024)	(.021)	(.020)	(.001)	.104	(.053)	(.055)	(.057)	(.010)
9	.013	(.016)	(.013)	(.013)	(.000)	.049	(.044)	(.046)	(.046)	(.005)
10+	.042	(.030)	(.027)	(.020)	(.001)	.229	(.193)	(.186)	(.181)	(.010)
Median	3	(2)	(2)	(2)	(2)	6	(5)	(5)	(5)	(4)

Instant Coffee

k^b	1					2				
0	.634	(.318)[k]	(.264)[ℓ]	(.244)[m]	(.874)[n]	.496	(.186)[k]	(.148)[ℓ]	(.134)[m]	(.295)[n]
1	.092	(.222)	(.258)	(.272)	(.117)	.104	(.156)	(.170)	(.177)	(.626)
2	.029	(.151)	(.170)	(.176)	(.006)	.046	(.128)	(.138)	(.142)	(.061)
3	.018	(.102)	(.110)	(.113)	(.002)	.018	(.104)	(.112)	(.103)	(.013)
4	.089	(.069)	(.071)	(.072)	(.001)	.084	(.084)	(.089)	(.092)	(.003)
5	.045	(.046)	(.046)	(.046)	(.000)	.044	(.068)	(.071)	(.072)	(.001)
6	.015	(.031)	(.030)	(.027)	(.000)	.020	(.054)	(.056)	(.056)	(.001)
7	.005	(.021)	(.018)	(.019)	(.000)	.013	(.044)	(.044)	(.045)	(.000)
8	.029	(.014)	(.012)	(.012)	(.000)	.045	(.035)	(.036)	(.036)	(.000)
9	.005	(.009)	(.008)	(.007)	(.000)	.014	(.028)	(.028)	(.029)	(.000)
10+	.039	(.017)	(.013)	(.012)	(.000)	.116	(.113)	(.108)	(.114)	(.000)
Median	0	(1)	(1)	(1)	(0)	1	(3)	(3)	(3)	(1)

[a] *Expected frequencies are shown in parentheses (one time-period = four weeks)*

[b] *One unit = 250 grams for regular coffee and margarine, and 50 grams for instant coffee*

[c] *Poisson ∧ Gamma (1.297, .515) = NBD*

[d] *Condensed-2 Poisson ∧ Gamma (1.297, .515) = Condensed-2 NBD*

[e] *Condensed-3 Poisson ∧ Gamma (1.297, .515) = Condensed-3 NBD*

[f] *IG ∧ Bivariate Natural Conjugate (.0402, 2.4097, 3.000001)*

[g] *Poisson ∧ Gamma (1.781, .412)*

[h] *Condensed-2 Poisson ∧ Gamma (1.781, .412)*

[i] *Condensed-3 Poisson ∧ Gamma (1.781, .412)*

[j] *IG ∧ Bivariate Natural Conjugate (.0232, 1.8765, 3.000001)*

[k] *Poisson ∧ Gamma (1.049, .495)*

[ℓ] *Condensed-2 Poisson ∧ Gamma (1.049, .495)*

[m] *Condensed-3 Poisson ∧ Gamma (1.049, .495)*

[n] *IG ∧ Bivariate Natural Conjugate (.00703, 6.99796, 3.000001)*

Table 4

Chi-Square Goodness-of-Fit Measures of All Four Models for Each Product Category

Product Class and Model	No. of time-periods[a]: 1 — χ^2 statistic	1 — Degrees of freedom	1 — Significance level	2 — χ^2 statistic	2 — Degrees of freedom	2 — Significance level
Margarine						
NBD	206.653	2	.000	350.421	5	.000
Condensed-2 NBD	386.940	2	.000	605.286	5	.000
Condensed-3 NBD	490.492	2	.000	1,312.057	5	.000
Compound IG	2,965.104	2	.000	4,596.896	5	.000
Regular Coffee						
NBD	137.640	2	.000	194.269	5	.000
Condensed-2 NBD	139.991	2	.000	388.233	5	.000
Condensed-3 NBD	391.891	2	.000	520.140	5	.000
Compound IG	9,964.416	2	.000	10,874.721	5	.000
Instant Coffee						
NBD	630.971	1	.000	810.734	1	.000
Condensed-2 NBD	949.294	1	.000	1,221.276	1	.000
Condensed-3 NBD	1,103.826	1	.000	1,434.294	1	.000
Compound IG	64,563.803	1	.000	28,269.984	1	.000

[a] *One time-period = four weeks*

Table 5

Simple Pairwise Correlations Between Relative Frequencies of Choice and Quantity Bought[a]

	M1	M2	M3	M4	M5			
Margarine	-.303[b,d]	-.061	.244[c]	-.069	-.168			
	RC1	RC2	RC3	RC4	RC5	RC6	RC7	RC8
Regular coffee	-.681[d]	-.418[c]	.154	-.661[d]	-.337	.049	-.207	.326
	IC1	IC2	IC3	IC4	IC5	IC6	IC7	IC8
Instant coffee	.016	-.181	-.591[d]	-.071	.244	.290	-.491[d]	.022

[a] *The correlations have been computed across groups of households. The unit quantity chosen was 1,000 grams for margarine and regular coffee and 500 grams for instant coffee. Eighty-eight groups have been formed for margarine, thirty-mine for regular coffee and thirty-seven for instant coffee. For margarine and regular coffee, Group i comprises the households who made between $(4i-4)$ and $(4i-1)$ purchases of unit quantity, the limits being included ($i=1,2,\ldots 38$ for regular coffee; $i=1,2,\ldots,87$ for margarine). For instant coffee, Group i comprises the households who made between $(10i-10)$ and $(10i-1)$ purchases of unit quantity ($i=1,2,\ldots,36$).*

[b] *The tests on the correlation coefficients are based on the z-transforms (Rao 1973, p. 433).*

[c] *Significant at $p < .05$.*

[d] *Significant at $p < .01$.*

Table 6

Empirical and Fitted Distributions of the Quantity of Use of a Major Brand in Each Product Class Over Two Time-Periods[a]

x^b	Brand MI						Brand RCI						Brand ICI					
	No. of time-periods						No. of time-periods						No. of time-periods					
	1			2			1			2			1			2		
0	.880	(.834)[c]	(.847)[d]	.849	(.784)[c]	(.787)[d]	.894	(.862)[e]	(.866)[f]	.839	(.822)[e]	(.818)[f]	.950	(.889)[g]	(.985)[h]	.920	(.850)[g]	(.913)[h]
1	.039	(.075)	(.108)	.037	(.074)	(.082)	.037	(.067)	(.084)	.052	(.066)	(.068)	.021	(.056)	(.014)	.036	(.059)	(.080)
2	.040	(.037)	(.034)	.041	(.040)	(.058)	.026	(.031)	(.042)	.034	(.035)	(.042)	.006	(.024)	(.001)	.009	(.030)	(.006)
3	.011	(.021)	(.007)	.012	(.027)	(.043)	.009	(.017)	(.006)	.012	(.024)	(.035)	-	(.013)	(.000)	.002	(.018)	(.001)
4	.011	(.013)	(.002)	.009	(.019)	(.018)	.020	(.010)	(.001)	.020	(.017)	(.025)	.016	(.007)	(.000)	.018	(.012)	(.000)
5	.002	(.008)	(.001)	.009	(.014)	(.007)	.007	(.006)	(.001)	.010	(.014)	(.008)	.001	(.004)	(.000)	.002	(.009)	(.000)
6	.008	(.005)	(.001)	.012	(.011)	(.003)	.004	(.004)	(.000)	.004	(.009)	(.003)	-	(.003)	(.000)	-	(.006)	(.000)
7	.001	(.003)	(.000)	.004	(.008)	(.001)	.002	(.002)	(.000)	.006	(.007)	(.001)	-	(.002)	(.000)	-	(.005)	(.000)
8	.004	(.002)	(.000)	.007	(.006)	(.001)	.001	(.001)	(.000)	.008	(.003)	(.000)	.004	(.001)	(.000)	.006	(.004)	(.000)
9	.000	(.001)	(.000)	.002	(.005)	(.000)	-	(.000)	(.000)	.002	(.002)	(.000)	-	(.001)	(.000)	-	(.003)	(.000)
10+	.004	(.001)	(.000)	.018	(.012)	(.000)	.000	(.000)	(.000)	.013	(.001)	(.000)	.002	(.000)	(.000)	.007	(.004)	(.000)
χ^2 Goodness-of-fit measure[i]																		
		(30.4)	(417.9)		(47.5)	(614.9)		(49.6)	(*[j])		(19.0)	(392.8)		(83.7)	(*[j])		(70.1)	(*[j])
		1	1		2	2		2	-		1	1		2	-		2	-

[a] *Expected frequencies are shown in parentheses (one time-period = four weeks)*

[b] *One unit = 250 grams for regular coffee and margarine, and 50 grams for instant coffee*

[c] *Poisson ∧ Gamma (1.297,.515) ∧ Beta (.0840,.464) ∧ Binomial distribution*

[d] *IG ∧ Bivariate natural conjugate (.0402, 2.4097, 3.000001) ∧ Beta (.0840,.464) ∧ Binomial distribution*

[e] *Poisson ∧ Gamma (1.781,.412) ∧ Beta (.0740,.595) ∧ Binomial distribution*

[f] *IG ∧ Bivariate natural conjugate (.0232, 1.8765, 3.000001) ∧ Beta (.0740,.595) ∧ Binomial distribution*

[g] *Poisson ∧ Gamma (1.049,.495) ∧ Beta (.0640,.477) ∧ Binomial distribution*

[h] *IG ∧ Bivariate natural conjugate (.00703, 6.99796, 3.000001) ∧ Beta (.0640,.477) ∧ Binomial distribution*

[i] *The number below the χ^2 statistic is the number of degrees of freedom*

[j] *The star sign (*) indicates that the χ^2 statistic could not be computed because of a lack of degrees of freedom*

Table 7

Actual and Theoretical Coefficients of Variation of the Brand Purchase Distributions Over Two Time-Periods[a]

Actual and theoretical coefficients of variation	Brand M1		Brand RC1		Brand IC1	
	No. of time-periods		No. of time-periods		No. of time-periods	
	1	2	1	2	1	2
Actual	3.671	3.632	3.550	3.055	5.979	5.316
NBD Λ BB[b]	3.084	2.883	3.399	3.150	3.798	3.529
Condensed-2 NBD Λ BB	2.955	2.811	3.257	3.072	3.629	3.433
Condensed-3 NBD Λ BB	2.909	2.787	3.207	3.044	3.569	3.400

a One time-period = four weeks

b NBD Λ BB indicates that the negative binomial distribution describes purchase timing and the beta binomial distribution describes brand choice.

Table 8

Actual and Fitted Cumulative Brand Penetrations
Within Each Product Class Over Several Time-Periods[a]

Time period[b]	Brand M1	Brand RC1	Brand IC1
1	12.0[c] (16.6)[d] [15.4][e]	10.6 (13.9)[f] [12.5][g]	5.0 (11.1)[h] [1.6][i]
7	22.2 (30.0) [28.5]	29.9 (25.7) [26.5]	16.9 (21.9) [17.7]
13	28.9 (32.8) [29.3]	40.1 (28.1) [32.0]	22.9 (26.6) [21.4]
20	35.0 (34.0) [35.2]	46.4 (29.2) [34.9]	27.7 (26.0) [24.5]
26	38.4 (34.8) [38.1]	52.0 (29.6) [36.0]	31.0 (26.7) [25.0]

[a] *In percentage*

[b] *One time-period = four weeks. One year is made of thirteen periods.*

[c] *Actual penetration*

[d] *Poisson* $\wedge$ *Gamma (1.297, .515)* $\wedge$ *Beta (.084, .464)* $\wedge$ *Binomial*

[e] *IG* $\wedge$ *Bivariate natural conjuguate (.0402, 2.4097, 3.000001)* $\wedge$ *Beta (.084, .464)* $\wedge$ *Binomial*

[f] *Poisson* $\wedge$ *Gamma (1.781, .412)* $\wedge$ *Beta (.074, .595)* $\wedge$ *Binomial*

[g] *IG* $\wedge$ *Bivariate natural conjuguate (.0232, 1.8765, 3.000001)* $\wedge$ *Beta (.074, .595)* $\wedge$ *Binomial*

[h] *Poisson* $\wedge$ *Gamma (1.049, .495)* $\wedge$ *Beta (.064, .477)* $\wedge$ *Binomial*

[i] *IG* $\wedge$ *Bivariate natural conjuguate (.00703, 6.99796, 3.000001)* $\wedge$ *Beta (.064, .477)* $\wedge$ *Binomial*

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